9. Věty o limitách posloupností

- ODVOZENÉ LIMITY: $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ $\quad \lim\limits_{n\to\infty} c = c$ ($c \in R$) konst.

- VĚTY:

Jestliže jsou posloupnosti $(a_n)_{n=1}^{\infty}$ a $(b_n)_{n=1}^{\infty}$ konvergentní (tj. mají limitu), potom jsou konvergentní i posloupnosti

$$(a_n + b_n)_{n=1}^{\infty},\ (a_n - b_n)_{n=1}^{\infty},\ (a_n \cdot b_n)_{n=1}^{\infty},\ \left(\frac{a_n}{b_n}\right)_{n=1}^{\infty}\ (b_n \neq 0).$$

Je-li $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$, pak

$$\lim_{n\to\infty}(a_n + b_n) = a + b \qquad \lim_{n\to\infty}(a_n \cdot b_n) = ab$$

$$\lim_{n\to\infty}(a_n - b_n) = a - b \qquad \lim_{n\to\infty}\frac{a_n}{b_n} = \frac{a}{b} \quad \begin{bmatrix} b_n \neq 0 \\ b \neq 0 \end{bmatrix}$$

$$\lim_{n\to\infty}(c a_n) = c \cdot a \quad (c \in R \text{ konst.})$$

[důkazy v učebnici]

Příklady

① Zjistěte, zda jsou dané posloupnosti konvergentní, příp. určete jejich limitu

a) $\left(\frac{1+5n}{n}\right)_{n=1}^{\infty}$ $\left[= \frac{(1+5n)_{n=1}^{\infty} \text{ div.}}{(n)_{n=1}^{\infty} \text{ div.}} \right\}$ upravit jinak $\Big]$

$$\underbrace{\left(\frac{1+5n}{n}\right)_{n=1}^{\infty} = \left(\frac{1}{n} + 5\right)_{n=1}^{\infty} = \underbrace{\underbrace{\left(\frac{1}{n}\right)_{n=1}^{\infty}}_{\text{konverg.}} + \underbrace{(5)_{n=1}^{\infty}}_{\text{konverg.}}}}_{\text{konverg.}}$$

$$\lim_{n\to\infty} \frac{1+5n}{n} = \lim_{n\to\infty}\left(\frac{1}{n} + 5\right) = \underbrace{\lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty} 5}_{\text{při zkráceném zápisu lze vynechat}} = 0 + 5 = \underline{5}$$

b) $$\underbrace{\left(\frac{3}{n^2}\right)_{n=1}^{\infty} = \left(3 \cdot \frac{1}{n} \cdot \frac{1}{n}\right)_{n=1}^{\infty}}_{\text{konv.}} = \underbrace{(3)_{n=1}^{\infty}}_{\text{konv.}} \cdot \underbrace{\left(\frac{1}{n}\right)_{n=1}^{\infty}}_{\text{konv.}} \cdot \underbrace{\left(\frac{1}{n}\right)_{n=1}^{\infty}}_{\text{konv.}}$$

$$\lim_{n\to\infty}\frac{3}{n^2} = \lim_{n\to\infty} 3 \cdot \lim_{n\to\infty}\frac{1}{n} \cdot \lim_{n\to\infty}\frac{1}{n} = 3 \cdot 0 \cdot 0 = \underline{0}$$

ZKRÁCENĚ

$$= \lim_{n\to\infty} 3 \cdot \lim_{n\to\infty}\frac{1}{n^2} = 3 \cdot 0 = \underline{0}$$

$\lim\limits_{n\to\infty}\frac{1}{n} = \lim\limits_{n\to\infty}\frac{1}{n^2} = 0$ (obd. i pro vyšší mocniny)

c) $\left(\frac{1+3n}{2n+1}\right)_{n=1}^{\infty}$ $\left[= \frac{(1+3n)_{n=1}^{\infty}}{(2n+1)_{n=1}^{\infty}} \begin{matrix} \to \text{není omez.} \to \text{diverg.} \\ \to \text{není omez.} \to \text{diverg.} \end{matrix} \right\} \text{ musíme upravit JINAK} \Big]$

$$\left(\frac{1+3n}{2n+1}\cdot\frac{\frac{1}{n}}{\frac{1}{n}}\right)_{n=1}^{\infty} = \left(\frac{\frac{1}{n}+3}{2+\frac{1}{n}}\right)_{n=1}^{\infty} = \frac{\left(\frac{1}{n}+3\right)_{n=1}^{\infty}}{\left(2+\frac{1}{n}\right)_{n=1}^{\infty}} \begin{matrix} \to \text{konv.} \\ \to \text{konv.} \end{matrix} \Big\} \text{ konv.}$$

$$\lim_{n\to\infty}\frac{1+3n}{2n+1} = \lim_{n\to\infty}\frac{\frac{1}{n}+3}{2+\frac{1}{n}} = \frac{\lim_{n\to\infty}\left(\frac{1}{n}+3\right)}{\lim_{n\to\infty}\left(2+\frac{1}{n}\right)} = \frac{0+3}{2+0} = \underset{\sim}{\frac{3}{2}}$$

② Určete (pokud ex.)

a) $\lim_{n\to\infty}\left(-\frac{1}{2n}-7\right) = \lim_{n\to\infty}\left(-\frac{1}{2}\cdot\frac{1}{n}-7\right) = \underset{[\text{konv.}]}{\lim_{n\to\infty}\left(-\frac{1}{2}\right)}\cdot\underset{[\text{konv.}]}{\lim_{n\to\infty}\frac{1}{n}} - \underset{[\text{konv.}]}{\lim_{n\to\infty} 7} =$

$$= -\frac{1}{2}\cdot 0 - 7 = \underset{\sim}{-7}$$

b) $\lim_{n\to\infty}\left(\frac{(-1)^n}{n^2}-2\right)$ $\left[= \underset{\downarrow\ \text{diverg.}}{\lim_{n\to\infty}(-1)^n}\cdot\underset{\downarrow\ \text{konv.}}{\lim_{n\to\infty}\frac{1}{n^2}} - \underset{\downarrow\ \text{konv.}}{\lim_{n\to\infty} 2} \quad \text{POZOR!} \right]$

$\left[\text{ALE } \lim_{n\to\infty}\frac{(-1)^n}{n} = 0\right]$ $\left[\text{viz minulý článek } \underset{\text{divergentní}}{\left((-1)^n\right)_{n=1}^{\infty}}, \text{ ALE } \underset{\text{KONVERG.}}{\left(\frac{(-1)^n}{n}\right)_{n=1}^{\infty}}\right]$

$$= \lim_{n\to\infty}\left(\frac{(-1)^n}{n}\cdot\frac{1}{n}-2\right) = \lim_{n\to\infty}\frac{(-1)^n}{n}\cdot\lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty} 2 =$$

$$= 0.0 - 2 = \underset{\sim}{-2}$$

c) $\lim_{n\to\infty}\left(\frac{2}{n}+\frac{(-1)^n}{n^2}\right) = \lim_{n\to\infty}\left(2\cdot\frac{1}{n}+\frac{(-1)^n}{n}\cdot\frac{1}{n}\right) = 2\cdot 0 + 0\cdot 0 = \underset{\sim}{0}$

③ Rozhodněte, které z posl. jsou konvergentní a příp. určete jejich limity

a) $\left(\frac{3n+1}{n+2}\right)_{n=1}^{\infty} = \left(\frac{3n+1}{n+2}:\frac{\frac{1}{n}}{\frac{1}{n}}\right)_{n=1}^{\infty} = \left(\frac{3+\frac{1}{n}}{1+\frac{2}{n}}\right)_{n=1}^{\infty}$ konv.

$$\lim_{n\to\infty}\frac{3n+1}{n+2} = \lim_{n\to\infty}\frac{3+\frac{1}{n}}{1+\frac{2}{n}} = \frac{3+0}{1+0} = \underset{\sim}{3}$$

$\left[\text{lze i } \lim_{n\to\infty}\frac{n\left(3+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)} = \lim_{n\to\infty}\frac{3+\frac{1}{n}}{1+\frac{2}{n}}, \text{ protože } \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{n}{n} = 1\right]$

b) $\left(\frac{n^2+1}{n^2+4}\right)_{n=1}^{\infty} = \left(\frac{n^2}{n^2}\cdot\frac{1+\frac{1}{n^2}}{1+\frac{4}{n^2}}\right)_{n=1}^{\infty} = \left(\frac{1+\frac{1}{n^2}}{1+\frac{4}{n^2}}\right)_{n=1}^{\infty}$ $\begin{matrix}\to \text{konv.}\\ \to \text{konv.}\end{matrix}\Big\}$ konv.

$$\lim_{n\to\infty}\frac{n^2+1}{n^2+4} = \lim_{n\to\infty}\frac{1+\frac{1}{n^2}}{1+\frac{4}{n^2}} = \frac{1+0}{1+0} = \underline{1}$$

POKUD PODÍL MNOHOČLENŮ => VYTKNEME V ČITAL. A JMENOV. NEJVYŠŠÍ MOCNINU $\underline{n}$ OBSAŽENOU VE JMENOVATELI !

c) $\left(\frac{2n}{n^2+2}\right)_{n=1}^{\infty} = \left(\frac{\frac{2}{n}}{1+\frac{2}{n^2}}\cdot\frac{n^2}{n^2}\right)_{n=1}^{\infty} = \left(\frac{\frac{2}{n}}{1+\frac{2}{n^2}}\right)_{n=1}^{\infty}$ $\begin{matrix}\to \text{konv.}\\ \to \text{konv.}\end{matrix}\Big\}$ konv.

$$\lim_{n\to\infty}\frac{2n}{n^2+2} = \lim_{n\to\infty}\frac{\frac{2}{n}}{1+\frac{2}{n^2}} = \frac{0}{1+0} = \frac{0}{1} = \underline{0}$$

d) $\left(\frac{n^3-4}{5n^2}\right)_{n=1}^{\infty} = \left(\frac{n-\frac{4}{n^2}}{5}\cdot\frac{n^2}{n^2}\right)_{n=1}^{\infty} = \left(\frac{\overset{\text{dir.}}{n}-\overset{\text{konv.}}{\frac{4}{n^2}}}{\underset{\text{konv.}}{5}}\right)_{n=1}^{\infty}$ => divergentní

$[(n)_{n=1}^{\infty}$ divergentní $]$

[kdyby: $\left(\frac{n^3-4}{5n^2}\cdot\frac{n^3}{n^3}\right)_{n=1}^{\infty} = \left(\frac{\overset{\text{konv.}}{1}-\overset{\text{konv.}}{\frac{4}{n^3}}}{\underset{\text{konv.}}{\frac{5}{n}}}\right)_{n=1}^{\infty}$ [=> diverg.]

!

$\lim_{n\to\infty}\frac{n^3-4}{5n^2} = \lim_{n\to\infty}\frac{1-\frac{4}{n^3}}{\frac{5}{n}} = \frac{1-0}{0}$! NELZE]

VYTÝKAT PROTO JEN NEJVYŠŠÍ MOCNINU n ve jmen. (ne v čitateli)

JMEN. $\neq 0$

e) $\left(\frac{5+(-1)^n\cdot n}{n}\right)_{n=1}^{\infty} = \left(\frac{\frac{5}{n}+(-1)^n}{1}\cdot\frac{n}{n}\right)_{n=1}^{\infty} = \frac{\overset{\text{konv.}}{\left(\frac{5}{n}\right)_{n=1}^{\infty}} + \overset{\text{dir.}}{\left((-1)^n\right)_{n=1}^{\infty}}}{\underset{\text{konv.}}{(1)_{n=1}^{\infty}}}$ => diverg.

[kdyby: $\left(\frac{n^2}{n^2}\cdot\frac{\overset{\text{konv.}}{\frac{5}{n^2}}+\overset{\text{konv.}}{\frac{(-1)^n}{n}}}{\frac{1}{n}}\right)_{n=1}^{\infty}$ => divergentní

konv., ale $\lim_{n\to\infty}\frac{1}{n}=0$, jmen. musí být $\neq 0$ => NELZE]

$\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$

$\lim_{n\to\infty}(-1)^n$ neex.

f) $\left(\frac{(n-3)(2n-1)}{(n+1)^2}\right)_{n=1}^{\infty} = \left(\frac{2n^2-n-6n+3}{n^2+2n+1}\right)_{n=1}^{\infty} = \left(\frac{2n^2-7n+3}{n^2+2n+1}\right)_{n=1}^{\infty} =$

$= \left(\frac{2-\frac{7}{n}+\frac{3}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}}\right)_{n=1}^{\infty}$ konvergentní

$$\lim_{n\to\infty}\frac{(n-3)(2n-1)}{(n+1)^2} = \lim_{n\to\infty}\frac{2-\frac{7}{n}+\frac{3}{n^2}}{1+\frac{2}{n}+\frac{1}{n^2}} = \frac{2-0+0}{1+0+0} = \underline{2}$$

g) DÚ $\left(\dfrac{3n^2+4n+5}{4n^3+2n^2}\right)_{n=1}^{\infty} = \left(\dfrac{\overset{\text{konv.}}{\frac{3}{n^3}} + \overset{\text{konv.}}{\frac{4}{n^2}} + \overset{\text{konv.}}{\frac{5}{n}}}{\underset{\text{konv.}}{4} + \underset{\text{konv.}}{\frac{2}{n}}}\right)_{n=1}^{\infty} \Rightarrow$ konv.

$$\lim_{n\to\infty} \frac{3n^2+4n+5}{4n^3+2n^2} = \lim_{n\to\infty}\left(\frac{\frac{3}{n}+\frac{4}{n^2}+\frac{5}{n^3}}{4+\frac{2}{n}}\cdot\frac{n^3}{n^3}\right) = \frac{0+0+0}{4+0} = \underline{0}$$

h) DÚ $\left(\dfrac{5n^2-4n+3}{3n^2+2n-1}\right)_{n=1}^{\infty}$

$$\lim_{n\to\infty}\left(\frac{5-\frac{4}{n}+\frac{3}{n^2}}{3+\frac{2}{n}-\frac{1}{n^2}}\cdot\frac{n^2}{n^2}\right) = \lim_{n\to\infty}\frac{5-\frac{4}{n}+\frac{3}{n^2}}{3+\frac{2}{n}-\frac{1}{n^2}} = \frac{5-0+0}{3+0-0} = \underline{\frac{5}{3}}$$

④ Vypočítejte

a) $$\lim_{n\to\infty}\frac{n!}{(n+1)!-n!} = \lim_{n\to\infty}\frac{n!}{(n+1)n!-n!} = \lim_{n\to\infty}\frac{n!}{n!(n+1-1)} =$$

$$= \lim_{n\to\infty}\frac{1}{n} = \underline{0}$$

b) DÚ $$\lim_{n\to\infty}\frac{(n+1)!}{n!-(n+1)!} = \lim_{n\to\infty}\frac{(n+1)n!}{n!-(n+1)n!} = \lim_{n\to\infty}\frac{(n+1)\cancel{n!}}{\cancel{n!}(1-(n+1))} =$$

$$= \lim_{n\to\infty}\frac{n+1}{1-n-1} = \lim_{n\to\infty}\frac{n+1}{-n} = \lim_{n\to\infty}\left(-1-\frac{1}{n}\right) = -1-0 = \underline{-1}$$

c) $$\lim_{n\to\infty}\frac{1+2+3+\ldots+n}{n^2} = \lim_{n\to\infty}\frac{\frac{n}{2}(1+n)}{n^2} = \lim_{n\to\infty}\frac{\frac{n}{2}+\frac{n^2}{2}}{n^2} =$$

$$\left[\begin{array}{l} \underbrace{1+2+3+\ldots+n}_{\text{součet } \underline{n} \text{ členů AP}} \\ s_n = \frac{n}{2}(a_1+a_n) = \frac{n}{2}(1+n) \end{array}\right] = \lim_{n\to\infty}\left(\frac{\frac{1}{2n}+\frac{1}{2}}{1}\cdot\frac{\cancel{n^2}}{\cancel{n^2}}\right) = \frac{0+\frac{1}{2}}{1} = \underline{\frac{1}{2}}$$

d) DÚ $$\lim_{n\to\infty}\frac{1+3+5+\ldots+(2n-1)}{2+4+6+\ldots+2n} = \lim_{n\to\infty}\frac{\frac{n}{2}(1+2n-1)}{\frac{n}{2}(2+2n)} = \lim_{n\to\infty}\frac{n^2}{n+n^2} =$$

$$\left[\begin{array}{ll} AP_1:\ \underset{a_1}{1}+3+5+\ldots+\underset{a_n}{(2n-1)} & AP_2:\ 2+4+6+\ldots+2n \\ s_n = \frac{n}{2}(a_1+a_n) \rightarrow a_n = a_1+(n-1)d \quad d=2 & a_1=2,\ a_n=2n \\ \qquad\qquad 2n-1 = 1+(n-1)2 & \text{počet členů} \ldots n \\ s_n = \frac{n}{2}(1+2n-1) \quad 2n-1 = 1+2n-2 & s_n = \frac{n}{2}(2+2n) = n+\frac{2n^2}{2} = n+n^2 \\ s_n = \frac{n}{\cancel{2}}\cdot\cancel{2}n = n^2 \quad \text{pl. (počet čl. } \underline{n}) & \end{array}\right]$$

$$= \lim_{n\to\infty}\frac{1}{\frac{1}{n}+1} = \frac{1}{0+1} = \underline{1}$$

⑤ Rozhodněte o konvergenci geometrických posloupností

a) $\left(\left(\frac{1}{2}\right)^n\right)_{n=1}^{\infty}$

$a_1 = \frac{1}{2},\ a_2 = \frac{1}{4},\ a_3 = \frac{1}{8}, \dots, a_8 = \frac{1}{256}, \dots \Rightarrow$

GP: $q = \frac{a_2}{a_1} = \frac{a_3}{a_4} = \dots = \frac{1}{2}$

$\left[\ q = \frac{a^n}{a^{n-1}} = \frac{\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^{n-1}} = \frac{\frac{1}{2}\cdot\left(\frac{1}{2}\right)^{n-1}}{\left(\frac{1}{2}\right)^{n-1}} = \frac{1}{2}\ \right]$

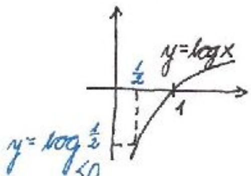
? $\lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$

$\forall \varepsilon > 0\ \exists n_0 \in N;\ n \geqq n_0 : |a_n - a| < \varepsilon$

$\left|\left(\frac{1}{2}\right)^n - 0\right| < \varepsilon$

$\left|\left(\frac{1}{2}\right)^n\right| < \varepsilon$

$\left(\frac{1}{2}\right)^n < \varepsilon$



$n \log\frac{1}{2} < \log\varepsilon \quad /: \underbrace{\log\frac{1}{2}}_{<0}$ (viz obr.)

$n \overset{!}{>} \frac{\log\varepsilon}{\log\frac{1}{2}}$

n_0 existuje pro lib. $\varepsilon > 0$

ZÁVĚR: $\lim\limits_{n\to\infty}\left(\frac{1}{2}\right)^n = 0 \qquad |q| < 1$

b) $(2^n)_{n=1}^{\infty}$

$a_1 = 2,\ a_2 = 4,\ a_3 = 8, \dots, a_8 = 256, \dots$

$q = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \dots = 2$

$\left[\ q = \frac{a_n}{a^{n-1}} = \frac{2^n}{2^{n-1}} = \frac{2\cdot 2^{n-1}}{2^{n-1}} = 2\ \right]$

divergentní $|q| > 1$

c) $(1^n)_{n=1}^{\infty}$

$a_1 = 1,\ a_2 = 1, \dots$

$q = 1 \quad \lim\limits_{n\to\infty} 1^n = 1$

konvergentní $q = 1$

d) $((-1)^n)_{n=1}^{\infty}$

$a_1 = -1,\ a_2 = 1,\ a_3 = -1, \dots$

divergentní $q = -1$

VĚTA

Každá geometrická posloupnost $(q^n)_{n=1}^{\infty}$ je pro

a) $|q| > 1$ divergentní

b) $q = -1$ divergentní

c) $q = 1$ konvergentní a $\lim\limits_{n\to\infty} 1^n = 1$

! d) $|q| < 1$ konvergentní a $\lim\limits_{n\to\infty} q^n = 0$

⑥ Rozhodněte o konvergenci

$a_1 = 4,\ q = -\frac{1}{2} \Rightarrow a_n = 4\cdot\left(-\frac{1}{2}\right)^{n-1} = 4\cdot\left(-\frac{1}{2}\right)^{-1}\cdot\left(-\frac{1}{2}\right)^n = 4\cdot\left(-\frac{2}{1}\right)^1\cdot\left(-\frac{1}{2}\right)^n =$

$= -8\cdot\left(-\frac{1}{2}\right)^n$

$\Rightarrow \lim\limits_{n\to\infty}\left(4\cdot\left(-\frac{1}{2}\right)^{n-1}\right) = \lim\limits_{n\to\infty}(-8)\left(-\frac{1}{2}\right)^n = \lim\limits_{n\to\infty}(-8)\cdot\lim\limits_{n\to\infty}\left(-\frac{1}{2}\right)^n = -8\cdot 0 = 0$

OBECNĚ: GP $\quad a_n = a_1 q^{n-1} \Rightarrow a_n = a_1 q^n \cdot q^{-1} = \frac{a_1}{q} q^n$

pro $|q| < 1$: $\lim\limits_{n\to\infty} a_1 q^{n-1} = \lim\limits_{n\to\infty}\frac{a_1}{q}q^n = \lim\limits_{n\to\infty}\underbrace{\frac{a_1}{q}}_{c\in R}\cdot\lim\limits_{n\to\infty} q^n = \frac{a_1}{q}\cdot 0 = 0$

VĚTA

Každá geometrická posloupnost $(a_n)_{n=1}^{\infty}$, pro jejíž kvocient q platí

a) $|q| < 1$ je konvergentní a $\lim\limits_{n\to\infty} a_n = 0$

b) $|q| > 1$ je divergentní

7. Rozhodnite o konvergenci

a) $(1{,}03^n)_{n=1}^{\infty}$ $a_1 = 1{,}03$ $a_2 = 1{,}0609$ $a_3 = 1{,}092727 \ldots$ $a_{100} = 19{,}21 \ldots$

DIVERGENTNÍ

$a_n = 1{,}03^n = 1{,}03^1 \cdot 1{,}03^{n-1} \Rightarrow q = 1{,}03 > 1$ divergentní

$(a_n = a_1 q^{n-1})$

b) $((0{,}6)^n)_{n=1}^{\infty}$ $a_1 = 0{,}6$ $a_2 = 0{,}36$ $a_3 = 0{,}216$ $\ldots$ $a_{100} = 6{,}5 \cdot 10^{-23}$

$\lim_{n\to\infty} 0{,}6^n = \underset{\sim}{0}$

GP: $|q| = |0{,}6| < 1$ konverg.

c) $(0{,}4^n + 3)_{n=1}^{\infty}$

$\lim_{n\to\infty} (0{,}4^n + 3) = \lim_{n\to\infty} 3 + \lim_{n\to\infty} 0{,}4^n = 3 + 0 = \underset{\sim}{3}$

GP: $q = 0{,}4$ $a_1 = 0{,}4$

$|q| < 1$ konverg.